Warm-up:

Critical Numbers 3.1 - 3.4 Practice WS
#1 - 5 and #10 - 14
Go over homework

p.187 HW Answers

17. a) $x = 3$
   b) \[ f'(x) \]
   \[ \frac{1}{3} \] \[ + \] \[ - \]
   incr $(3, \infty)$
   decr $(-\infty, 3)$
   c) Min $(3, -9)$
   d) \[ \frac{1}{3} \] \[ + \]

19. a) $x = 1$
   b) \[ f'(x) \]
   \[ + \] \[ - \] \[ - \] \[ + \]
   incr $(-\infty, 1)$
   decr $(1, \infty)$
   c) Max $(1, 5)$
   d) \[ \frac{1}{3} \] \[ + \]

25. a) $x = 1, -1$
   b) \[ f'(x) \]
   \[ + \] \[ - \] \[ + \]
   incr $(-\infty, -1)(1, \infty)$
   decr $(-1, 1)$
   c) Max $(-1, \frac{4}{3})$
   min $(1, -\frac{4}{3})$
   d) \[ \frac{1}{3} \] \[ + \]
27. a) $x = 0$
   (undefined but a critical #)
   b) $\begin{array}{c}
   + \\
   - \\
   0 \\
   + \\
   \end{array}$
   incr. $(-\infty, \infty)$
   c) no min/max
   d) $
   \longrightarrow
   $

55

57.

65. $g'(0) < 0$
67. $g'(-6) < 0$
69. $g'(0) > 0$
In groups of 3:

1. Each person gets one problem. Find the derivative.
2. Rotate papers. Identify all critical numbers.
3. Rotate papers. Complete the number line test.
4. Rotate papers back to original person. Identify increasing and decreasing intervals.
3.4 Concavity & the Second Derivative Test

Definition of Concavity

Let $f$ be differentiable on an open interval, $I$. The graph of $f$ is concave upward on $I$ if $f'$ is increasing on the interval and concave downward if $f'$ is decreasing on the interval.

Ex 1 Graph $f(x) = \frac{1}{3}x^3 - x$ and $f'(x)$.

$f(x) = \frac{1}{3}x^3 - x$

$f'(x) = x^2 - 1$

$f''(x) = 2x$

Q1: When is $f(x)$ increasing? What is the shape of $f(x)$ on that interval?

$(0, \infty)$

concave up

Q2: When is $f(x)$ decreasing? What is the shape of $f(x)$ on that interval?

$(-\infty, 0)$

concave down

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concave down</td>
<td>decreasing</td>
<td>negative (below x-axis)</td>
</tr>
<tr>
<td>Concave up</td>
<td>increasing</td>
<td>positive (above x-axis)</td>
</tr>
</tbody>
</table>

(-\infty, 0) | Concave down  | decreasing |
(0, \infty)  | Concave up    | increasing |
Test for Concavity

Let \( f \) be a function whose second derivative exists on an open interval \( I \) and \( f''(x) > 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave upward in \( I \).

1. If \( f''(x) > 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave upward in \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in \( I \), then the graph of \( f \) is concave downward in \( I \).

Ex 2 Determine the open intervals on which \( f(x) = 6(x^2 + 3)^{-1} \) is concave up or concave down.

\[
f(x) = 6(x^2 + 3)^{-1}
\]

1. \( f'(x) = -12(x^2 + 3)^{-2}(2x) = \frac{-12x}{(x^2 + 3)^2} \)

2. \( f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4} = \frac{-12(x^2 + 3) + 48x^2}{(x^2 + 3)^3} = \frac{-12(x^2 + 3) + 48x^2}{(x^2 + 3)^3} = \frac{-12x^2 - 36 + 48x^2}{(x^2 + 3)^3} = \frac{36x^2 - 36}{(x^2 + 3)^3} \)

\[\text{will never be undefined}\]

3. Set \( f''(x) = 0 \)

\[
36(x^2 - 1) = 0
\]

\[
36(x-1)(x+1) = 0
\]

\[x = 1, -1\]

\[
\begin{array}{cccccc}
\text{f''(x)} & + & - & + & \text{f''(x)} & + \\
\hline
-1 & | & | & | & 1 & \text{U}
\end{array}
\]

4. Concave up \((-\infty, -1) \cup (1, \infty)\)

5. Concave down \((-1, 1)\)
Definition of a Point of Inflection

Let $f$ be continuous on an open interval and let $c$ be a point in that interval. If the graph of $f$ has a tangent line at the point $(c, f(c))$, then this point is a point of inflection of the graph of $f$ if the concavity of $f$ changes from upward to downward (or downward to upward) at the point.

Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of $f$, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Ex 4 Determine the point(s) of inflection and discuss the concavity of $f(x) = x^4 - 4x^3$.

1. $f'(x) = 4x^3 - 12x^2$
2. $f''(x) = 12x^2 - 24x$
3. $12x^2 - 24x = 0$
   \[12x(x - 2) = 0\]
   \[x = 0, 2\]
4. $f''(x)$ graph with critical points at $x = 0, 2$
5. Concavity $\uparrow$ on $(-\infty, 0) (2, \infty)$
   $\downarrow$ on $(0, 2)$
6. P.O.I. $(0, 0)$
   $(2, -16)$
   $f(0) = 0$
   $f(2) = -16$
Ex 5 Determine the point(s) of inflection and discuss the concavity of $f(x) = x^4$.

1. $f'(x) = 4x^3$
2. $f''(x) = 12x^2$
3. $12x^2 = 0$
   $x = 0$
4. $f''(x)$
   \[ \begin{array}{c|c|c|c}
   x & - & + & + \\
   \hline
   f'' & - & 0 & +
   \end{array} \]

5. $cc \uparrow (-\infty, 0) \downarrow (0, \infty)$

6. P.O.I. none
   \[ \text{blc concavity doesn't change} \]
Second Derivative Test

Let \( f \) be a function such that \( f'(x) = 0 \) and the second derivative of \( f \) exists on an open interval containing \( c \).

1. If \( f''(x) > 0 \), then \( f \) has a relative minimum at \((c, f(c))\).
2. If \( f''(x) < 0 \), then \( f \) has a relative maximum at \((c, f(c))\).

If \( f''(x) = 0 \), the test fails. That is, \( f \) may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Ex. 6 Use the Second Derivative Test to find all relative extrema of \( f(x) = -3x^3 + 5x^3 \).

1. Find critical \#s from \textbf{first deriv.}
   \[ f'(x) = -15x^4 + 15x^2 = 0 \]
   \[ -15x^2(x^2-1) = 0 \]
   \[ -15x^2(x-1)(x+1) = 0 \]
   \[ x = 0, 1, -1 \]

2. Plug in critical \#s to \textbf{\( f''(x) \)}
   \[ f''(0) = 0 \]
   \[ f''(1) = -30 < 0 \]
   \[ f''(-1) = 30 > 0 \]

3. At \( x = 0 \), 2nd deriv. test failed so do 1st deriv. test
   \[ f''(x) \]
   \[ \begin{array}{c|c|c|c|c|c}
   x & - & + & + & - \\
   \hline
   f'' & -1 & 0 & 1 & 0 & 1
   \end{array} \]
   \[ \text{neither min, nor max} \]

4. Is \( x = 0 \) a PD? \[ f''(x) \]
   \[ \begin{array}{c|c|c}
   x & + & - \\
   \hline
   f'' & 0 & 0
   \end{array} \]
   \[ \text{yes PD at } (0, \frac{2}{3}) \]

Max \((1, \frac{2}{3})\) f(t) plug into original to get y-value
Min \((-1, \frac{2}{3})\)